

# Chapter 4

## Random Field Generation

### 4.1 Introduction

The aim of this document is to explain the process of generating certain types of random fields. The purpose of such an exercise is that such fields can be very useful in climate simulation as driving forces. Our plan is to define random variables, then random fields. Such efforts will help us in the implementation of the results in later sections.

## 4.2 Random Variables

Let  $X$  be a variable that is generated from some kind of random process, such as flipping a coin, in which case  $X$  would take on the possible values of 0 and 1 with equal probability. We could also imagine a process where  $X$  takes on continuous values over some range, for example, the heights of individuals drawn from a very large population. Such a process as this leads us to a continuous *probability density function* or *pdf*,  $P(x)$ . The probability of  $X$  taking on a value in a given drawing lying between  $x$  and  $x + dx$  is given by  $P(x) dx$ . We call the value taken on by the *random variable*,  $X$ , a *realization* of  $X$ . Each drawing leads to a new realization of the random variable  $X$ .

Some useful properties of the random variable are its *mean value*, denoted by  $\mu$ , and given by

$$\mu \equiv \int_a^b xP(x) dx \equiv \langle x \rangle \quad (4.1)$$

where  $a$  and  $b$  denote the domain over which  $X$  can take on values. Integrating a variable or a function of the variable, say  $f(x)$  is called the *expectation value* of  $f(x)$  and is denoted

$$\langle f(x) \rangle \equiv \int_a^b f(x)P(x) dx \quad (4.2)$$

Another familiar property of a random variable is its variance defined by

$$\text{var}(x) \equiv \int_a^b (x - \mu)^2 P(x) dx \quad (4.3)$$

In this chapter we will make use of the normal distribution (sometimes called the Gaussian Distribution) often. It is the familiar bell shaped curve:

$$\mathcal{N}(x) \equiv \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x - \mu)^2}{2\sigma^2}\right) \quad (4.4)$$

It can be shown that the mean of the normal distribution is  $\mu$  and the variance is  $\sigma^2$ . The domain for  $x$  is over the infinite range,  $-\infty < x < \infty$ . The integral of  $\mathcal{N}(x)$  over the whole range is unity. If a variable is distributed according to the normal distribution with mean  $\mu$  and variance  $\sigma^2$  we indicate this by

$$X \sim \mathcal{N}(\mu, \sigma^2) \quad (4.5)$$

An important property of normally distributed variables is that if we add two such variables, the sum is also normally distributed such that  $\mu = \mu_1 + \mu_2$  and  $\sigma^2 = \sigma_1^2 + \sigma_2^2$ .

### 4.3 Random Functions

Next consider a well behaved function,  $G(y)$  that is a function defined over a range of values of its independent variable  $y$ , but from one realization to another the function changes. An example of such a random function is

$$G(y) = 4 + a \sin 3y + (1 + b)y^2 \quad (4.6)$$

where  $a$  and  $b$  are random variables  $\sim \mathcal{N}(0, 1)$ . We can specify that these two random variables are also *independent*. That is, the outcome of one of them does not depend in any way on the outcome of the other for a particular realization. Figure 1 shows 5 realizations of  $G(y)$ . We can compute the mean and variance of  $G(y)$ :

$$\langle G(y) \rangle = 4 + \langle a \rangle \sin 3y + \langle (1 + b)y^2 \rangle \quad (4.7)$$

This yields

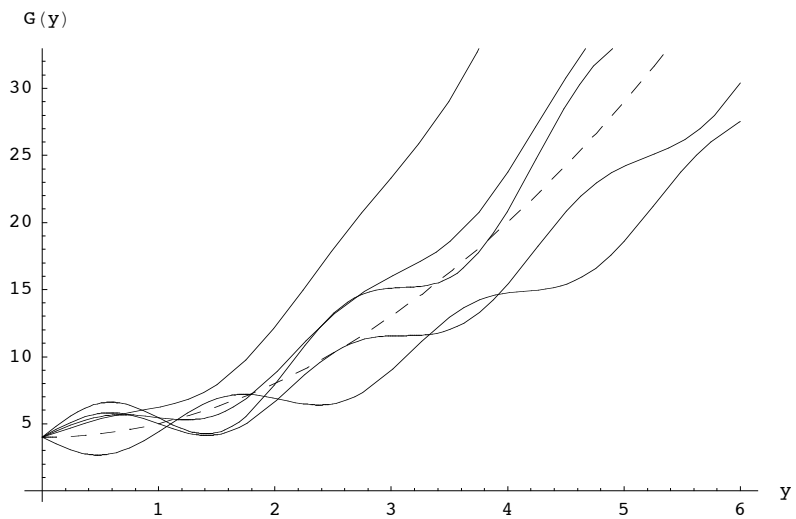
$$\langle G(y) \rangle = 4 + y^2 \quad (4.8)$$

the variance of  $G(y)$  is

$$\text{var}(G(y)) = y^2 \quad (4.9)$$

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## 4.4 Random Function on a Circular Rim

Consider next a real-valued function defined on the rim of a circle,  $f(\theta)$ ,  $0 \leq \theta \leq 2\pi$ . Such a function can be expanded into a Fourier Series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \quad (4.10)$$

The complex coefficients  $f_n$  can be computed by multiplying through the last equation by  $e^{-im\theta}$  and integrating from 0 to  $2\pi$ :

$$\int_0^{2\pi} e^{-im\theta} f(\theta) d\theta = \sum_{n=-\infty}^{\infty} f_n \int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi \sum_{n=-\infty}^{\infty} f_n \delta_{mn} \quad (4.11)$$

Finally,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta \quad (4.12)$$

The coefficients  $f_n = f_n^r + i f_n^i$  are complex, even though  $f(\theta)$  is real-valued.

Our next step is to make  $f(\theta)$  a random function. We will do this by making the coefficients  $f_n$  random variables. This means that both  $f_n^r$  and  $f_n^i$  are each random variables.

We have one further property we wish to attribute to  $f(\theta)$ . We want it to have mean zero and to have its statistics be *rotationally invariant* on the

circle. By this we mean that

$$\langle f(\theta) \rangle = 0, \quad (4.13)$$

and

$$\langle f(\theta)f(\theta') \rangle = g(|\theta - \theta'|) \quad (4.14)$$

Where  $g(\phi)$  is some given (real) function that we can construct later to our choosing. This latter means that the *correlation* of  $f$  evaluated at two different points on the circle depends only on the magnitude of the angle between the two points,  $|\theta - \theta'|$ . There is no preferred origin on the circle – the statistics are dependent only on relative positions. This rotational symmetry property imposes remarkable constraints on the coefficients  $f_n$  as we will see. First note that the *lagged correlation function*  $g(|\theta - \theta'|)$  is an even function of  $\theta - \theta'$  and so it can be expanded into a cosine Fourier Series:

$$g(\phi) = \frac{g_0}{2} + \sum_{l=1}^{\infty} g_l \cos l\phi \quad (4.15)$$

with

$$g_l = \frac{1}{\pi} \int_0^{2\pi} g(\phi) \cos l\phi d\phi \quad (4.16)$$

Now since  $f(\theta)$  is real, we can write

$$f_n^r = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \quad (4.17)$$

and

$$f_n^i = -\frac{1}{2\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \quad (4.18)$$

Next consider the *covariance* of the coefficients  $f_n^r$  and  $f_m^r$ :

$$\langle f_n^r f_m^r \rangle = \left( \frac{1}{2\pi} \right)^2 \int \int \langle f(\theta) f(\theta') \rangle \cos n\theta \cos m\theta' \, d\theta \, d\theta' \quad (4.19)$$

The bracketed quantity in the integral may be replaced by  $g(|\theta - \theta'|)$  and its Fourier Cosine Series. When this is done it is found that the integral vanishes unless  $l = m = n$  in which case it equals  $\pi^2$ . Finally,

$$\langle f_n^r f_m^r \rangle = \frac{1}{4} g_n \delta_{nm} \quad (4.20)$$

We can also show that the same is true for the imaginary parts of  $f_n$ :

$$\langle f_n^i f_m^i \rangle = \frac{1}{4} g_n \delta_{nm} \quad (4.21)$$

Moreover, there is no covariance between the real and imaginary parts:

$$\langle f_n^i f_m^r \rangle = 0 \quad (4.22)$$

We can see then that:

$$\langle |f_n|^2 \rangle = \langle (f_n^r)^2 \rangle + \langle (f_n^i)^2 \rangle = \frac{g_n}{2} \quad (4.23)$$

So the variance of the  $n^{\text{th}}$  Fourier Coefficient is just  $g_n$ , the  $n^{\text{th}}$  Fourier Coefficient of the lagged covariance.

Our last step is to examine the integral over the circle of the variance of  $f(\theta)$ :

$$\int_0^{2\pi} \langle f^2(\theta) \rangle d\theta = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} e^{i(m-n)\theta} \langle f_m f_n^* \rangle d\theta \quad (4.24)$$

After performing the integral (giving  $2\pi\delta_{mn}$ ) the double sum collapses to:

$$\int_0^{2\pi} \langle f^2(\theta) \rangle d\theta = \sum_{n=-\infty}^{\infty} \langle |f_n|^2 \rangle = \frac{g_0}{2} + \sum_{n=1}^{\infty} g_n \quad (4.25)$$

This last effectively decomposes the integrated variance into a sum of contributions from the squares of each Fourier component. This allows us to see how much variance is contributed by each *wave number*  $n$  in the decomposition. In some contexts the variances are referred to as the *power* or *energy* in each frequency component. For example, if  $f(\theta)$  were a velocity component, its square would represent kinetic energy.

## 4.5 Spectral Decomposition

For all random functions whose statistics are normally distributed and homogeneous (rotationally invariant) on the circle, only one function is necessary

to completely describe the the random function, the *autocovariance function*:  $g(|\theta - \theta'|) \equiv \langle f(\theta)f(\theta') \rangle$ . But this is equivalent to knowing the Fourier Coefficients  $g_l, l = 0, 1, 2, \dots$ . A bar graph of  $g_l$  is a display of the characteristics of the function. This is called the *spectrum* for the random process  $f(\theta)$ . Basically, the spectrum tells us how the variance is distributed across the wave numbers. More variance concentrated in the small values of  $l$  indicates that the variability will be dominated by the larger scales.

## 4.6 On the Sphere

Now we move to the surface of the sphere. Let our function be defined by  $f(\hat{\mathbf{r}})$ . We want it also to be normally distributed. We can expand it into spherical harmonics:

$$f(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{nm} Y_{nm}(\hat{\mathbf{r}}) \quad (4.26)$$

This time we specify that the random function have statistics that are rotationally invariant on the sphere. The way to do this is to have its covariance depend only on the great circle distance between the two points  $\hat{\mathbf{r}}$  and  $\hat{\mathbf{r}}'$ . In other words the covariance will depend only on  $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'$ , which is the cosine of the opening angle between the two points on the sphere.

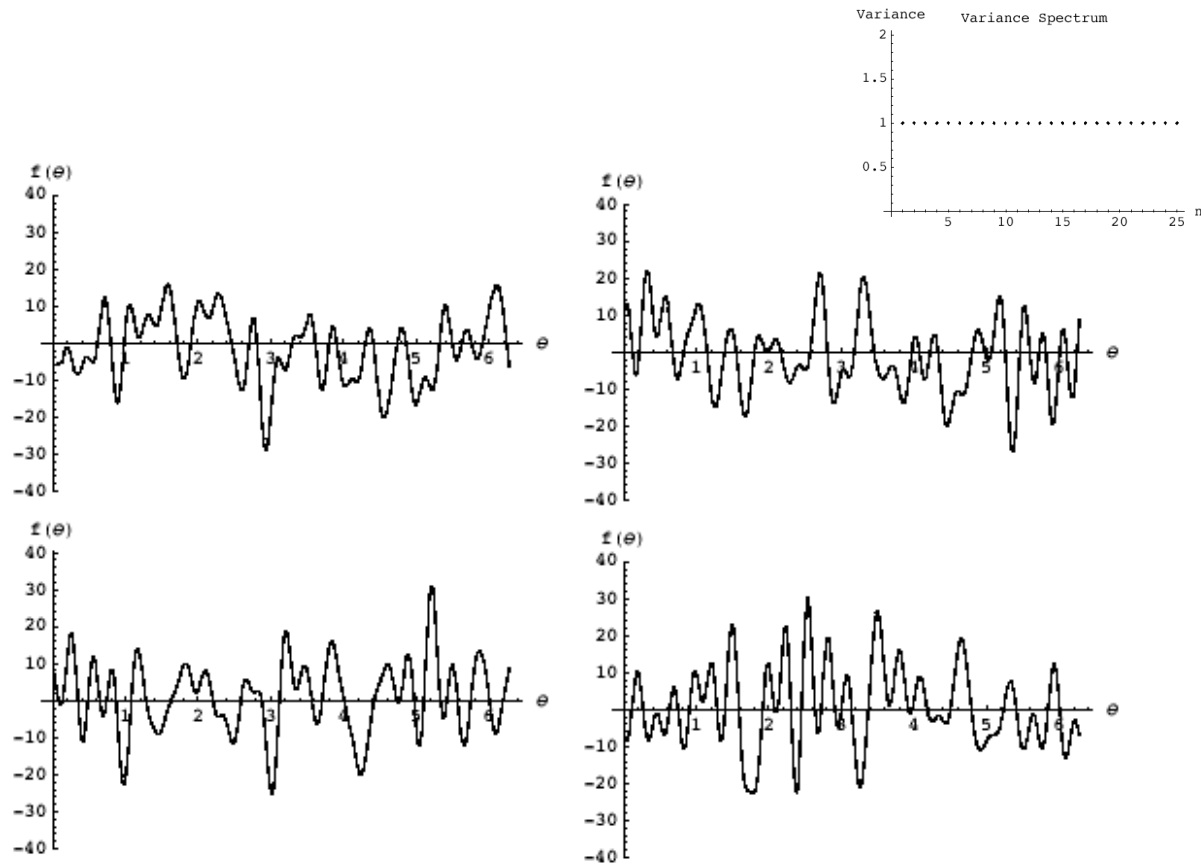


Figure 4.2: Four Realizations of the random function  $\sum_{n=-25}^{25} c_n e^{in\theta}$  where the  $c_n$  are complex random variables with real and imaginary parts  $\sim \mathcal{N}(0, 1)$ . Also  $c_n^* = c_{-n}$ . The power spectrum is shown in the upper right hand corner.

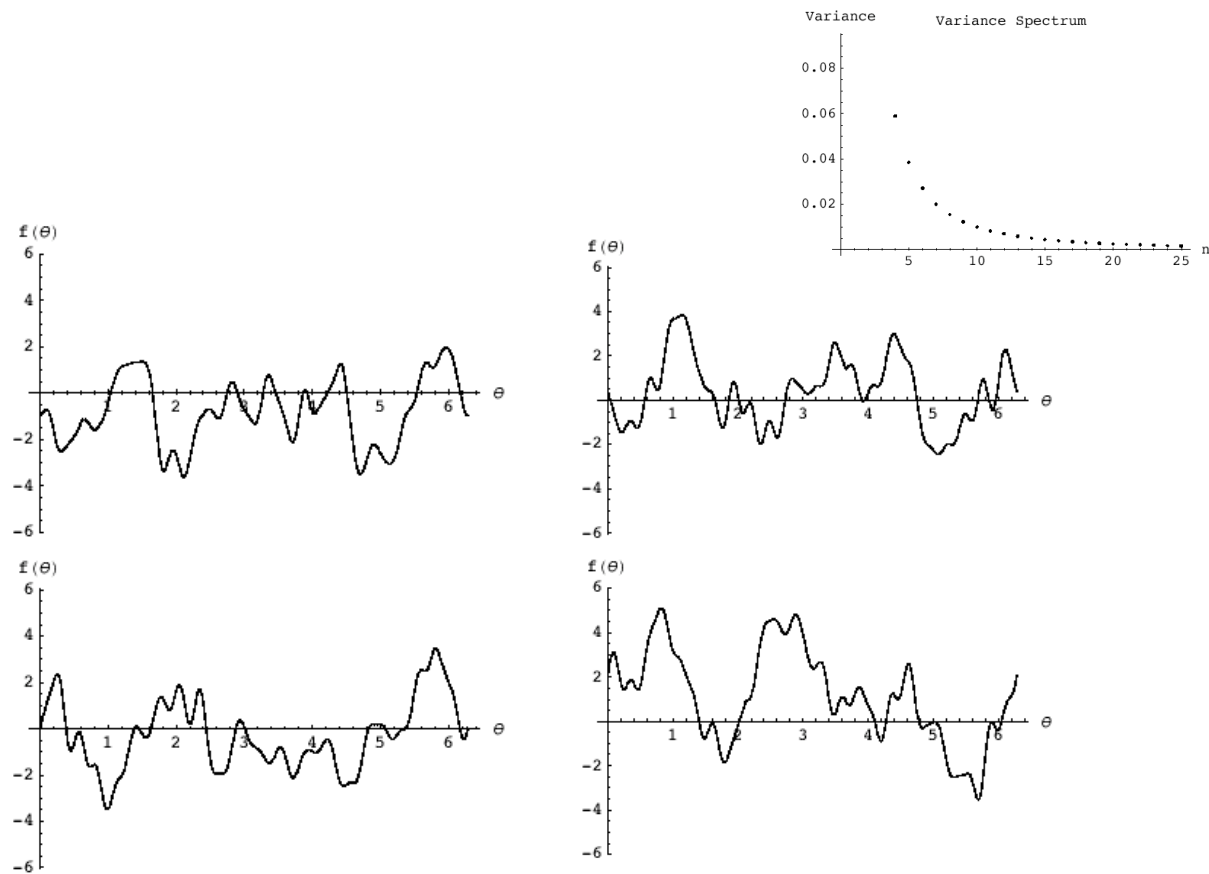


Figure 4.3: Four Realizations of the random function  $\sum_{n=-25}^{25} c_n e^{in\theta}$  where the  $c_n$  are complex random variables with real and imaginary parts  $\sim \mathcal{N}(0, \frac{1}{1+n^2})$ . Also  $c_n^* = c_{-n}$ . The power spectrum is shown in the upper right hand corner.

Consider now the covariance of  $f$  taken between the two points:

$$K(\hat{\mathbf{r}}, \hat{\mathbf{r}}') = K(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \quad (4.27)$$

This latter form is a perfect candidate for expansion in a Fourier Legendre Series;

$$K(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \sum_{n=0}^{\infty} K_n P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \quad (4.28)$$

Now we can use the addition theorem for spherical harmonics to obtain:

$$K(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \sum_{m=-n}^n K_n Y_{mn}^*(\hat{\mathbf{r}}) Y_{mn}(\hat{\mathbf{r}}') \quad (4.29)$$

This formula will prove useful presently.

Next expand our real *random field*  $f(\hat{\mathbf{r}})$  into spherical harmonics:

$$f(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{nm} Y_{nm}(\hat{\mathbf{r}}) \quad (4.30)$$

with

$$f_{nm} = \int \int Y_{nm}^*(\hat{\mathbf{r}}) f(\hat{\mathbf{r}}) d\Omega \quad (4.31)$$

Following the same steps as on the circle we examine:

$$\langle f_{nm}^* f_{n'm'} \rangle = \int \int \int \int Y_{mn}^*(\hat{\mathbf{r}}) Y_{n'm'}(\hat{\mathbf{r}}') \langle f(\hat{\mathbf{r}}) f(\hat{\mathbf{r}}') \rangle d\Omega d\Omega' \quad (4.32)$$

Next we substitute from the covariance expression above and find that the integrals can be performed yielding Kronecker Deltas.

$$\langle f_{nm}^* f_{n'm'} \rangle = \frac{4\pi \delta_{nn'} \delta_{mm'}}{2n+1} K_n = \langle |f_{nm}|^2 \rangle \quad (4.33)$$

This latter is called the spherical harmonic degree spectrum. Notice that the *power* in each spherical harmonic component is independent of the longitudinal wave number,  $m$ . So the variance is apportioned equally in each  $m$  component, but it changes from degree  $n$  to the next.